

Quantum mechanical histories and the Berry phase

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Abstract

We elaborate on the distinction between geometric and dynamical phase in quantum theory and we show that the former is intrinsically linked to the quantum mechanical probabilistic structure. In particular, we examine the appearance of the Berry phase in the consistent histories scheme and establish that it is the basic building block of the decoherence functional. These results are consequences of the novel temporal structure of histories-based theories.

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1 Introduction

The consistent histories formulation [1, 2, 3, 4] of quantum mechanics focuses on the temporally ordered properties of physical systems: these are known as histories. There are two important structural features of quantum mechanical history theories. The first is that there does not exist a probability measure in the space of all histories: there exists interference between pairs of histories.

The second is their non-trivial temporal structure, that allows a differentiation between the kinematical and the dynamical aspects of time. This is present in the quantum temporal logic formulation of consistent histories [5]; this allows a description of continuous-time histories [6, 7], in which there exist distinct generators of time translation according to whether they refer to dynamical or kinematical features of the histories [8, 9].

In this paper, we establish that this distinction is mirrored in a differentiation that is well known in standard quantum theory: the one between the dynamical phase due to Hamiltonian evolution and the geometric phase of Berry [10]. What is more, the geometric phase manifests itself strongly in the probabilistic structure of histories: it is the basic building block of the interference phase between pairs of histories. These results are established in an elementary fashion by the study of fine-grained, continuous-time histories; they can be then suitably generalised.

Now, the appearance (and measurability) of the geometric phase in the time evolution of quantum systems is arguably one of the most important structural features of quantum theory. Berry showed that when a system undergoes a cyclic evolution, due to an adiabatic change of parameters in the Hamiltonian, a contribution in the phase appears, that is purely geometric. In particular, the phase contribution does not depend on the details of the dynamics but only on the loop that was transversed by the system in the parameter space.

It was soon realised [11] that the Berry phase is the holonomy of a $U(1)$ connection on the parameter space. In fact it can be generalised for any kind of unitary evolution on the Hilbert space, since it arises by a natural $U(1)$ connection on the projective Hilbert space.

The geometric phase is a measurable quantity, that does not formally correspond to a self-adjoint operator. Furthermore it provides a paradigm and a motivation for investigations of topological phenomena in quantum theory, as it highlights the natural appearance of gauge structures in the quantum formalism.

The key point of Berry's result however, that sets the subject in the foundations of quantum theory, is the following: the Berry phase *has no analogue in the language of probability theory*.

A probabilistic theory for a physical system—either classical or quantum—has as basic notions *observables, propositions and states* that are represented by suitable mathematical objects. In classical probability theory observables are functions on a space Ω , propositions correspond to *measurable subsets* of Ω and states to probability distributions. In quantum theory these probabilistic

concepts are also fundamental: they are represented by Hilbert space objects. These are self-adjoint operators for observables, projection operators for propositions and density matrices for states.

However, the standard quantum mechanical formalism refers to properties of the system *at a single moment of time*: it assigns probabilities to possible instantaneous events and studies the evolution of these single-time probabilities. In this context, the phase of a Hilbert space vector is not physically relevant, as it does not enter the single-time probability assignment. When this phase is ignored, quantum theory is *only* a generalisation of probability theory, with main difference the non-distributivity of the lattice of propositions or equivalently the non-commutativity of the algebra of observables.

This is the attitude taken by approaches to quantum theory that attempt to write an axiomatic framework without assuming *a priori* the existence of a Hilbert space, for example, the C^* -approach, quantum logic schemes or the operational approach to quantum theory.

The existence of the Berry phase, as a measurable quantity, shows that the *single-time* probabilistic description does not exhaust the physical content of quantum theory. The Berry phase appears in distinction to the well-known phases of unitary evolution that are generated by a Hamiltonian. Its nature is purely kinematical and it is a manifestation of the non-trivial topology of the space of pure quantum states, since it appears naturally when we view the Hilbert space H of quantum theory as a complex line bundle over the projective Hilbert space PH [11, 12, 13]. The Berry phase is then the holonomy of the *natural* connection of this bundle (i.e. the connection induced by the inner product).

It needs to be emphasised that this bundle structure is irrelevant *to any probabilistic aspects of quantum theory*. In other words, in the unitary time evolution of quantum theory there appears an extra phase due to the topological structure of the theory; it has no intuitive physical explanation and it has no classical analogue—either in classical mechanics¹ or in classical probability theory.

In the single-time description of a quantum system the geometric phase is lost. Hence it is rather difficult to understand its physical meaning in standard quantum theory. However, the importance of geometric phase is more clear in a quantum theory that is based on *histories*. A history is defined at different moments of time, in distinction to standard single-time quantum theory. Such a formulation is provided by the consistent histories approach to quantum theory.

This approach was developed as a realist interpretational scheme for quantum theory [1, 2, 3, 4]. As such, it suffers from the generic problems of such schemes, i.e., contextuality of predictions about properties of the physical system [15]. Nonetheless, it provides a new insight in understanding the appearance of

¹The Hannay angle [14] is an analogue in classical mechanics. But this appears whenever certain degrees of freedom can be ignored due a symmetry, whereas the wave function is assumed to give a complete description of the quantum system.

the Berry phase in quantum theory, in a manner independent of any particular interpretational scheme one may choose to employ.

The basic object of the histories formalism is a history, i.e., a sequence of time-ordered propositions about properties of the physical system. It corresponds to different possible scenarios of the system. The main feature, that distinguishes quantum mechanical histories from the ones appearing in the theory of stochastic processes (classical probability theory), is that the probabilities for histories *do not satisfy the additivity condition*.

$$p(\alpha \vee \beta) = p(\alpha) + p(\beta), \quad (1)$$

where α and β are mutually exclusive scenarios. This is due to the fact that quantum theory is based on amplitudes rather than probability measures, and it further implies the existence of interference between histories.

The corresponding information, together with the probabilities, is encoded in an object called the *decoherence functional*. This object incorporates the kinematics, the dynamics and the initial condition of the physical system.

In our effort to identify the role of the Berry phase in the histories scheme, we arrived at a surprisingly simple result: the geometric phase *is the main building block of the decoherence functional*. Hence, interference between histories is ultimately to be attributed to the presence of the geometric phase. Moreover, we showed that the distinction between geometric phase and the dynamical phase of canonical quantum theory –i.e the one appearing due to Hamiltonian time evolution– is a manifestation of the temporal structure of history theories: the existence of two laws of time transformation each corresponding to the causal/kinematical and dynamical notions of time [8, 9].

2 The geometric phase

The simplest way to demonstrate the origin of the Berry phase is in the context of differential geometry. To this end, let us take the complex Hilbert space H to be finite dimensional ($H = \mathbf{C}^{n+1}$). The inner product $\langle z|w \rangle = \bar{z}_a w^a$ gives a metric $ds^2 = d\bar{z}_a dz^a$ (where a runs from 0 to n and z refers to coordinates with respect to a basis), from its real part, and a symplectic form $\omega = d\bar{z}_a \wedge dz^a$ on H , from its imaginary part.

The metric induces the standard metric to the unit sphere S^{2n+1} of all normalised vectors. The unit sphere is a $U(1)$ principal bundle over the projective Hilbert space PH , the space of rays; this structure is known as the Hopf bundle. An element of PH is represented by $[\psi]$, the equivalence class of all normalised vectors that differ from the normalised vector $|\psi\rangle$ only with respect to a phase. The metric on S^{2n+1} induces a metric on PH , defined as

$$ds^2(PH) = \frac{1}{1 + \bar{w}_a w^a} d\bar{w}_a dw^a \quad (2)$$

and an one-form $A = i\bar{w}_a dw^a$. Here, we have defined coordinates such that for $1 \leq a \leq n$, $w^a = z^a/z^0$.

In particular, the one-form A is a $U(1)$ connection form for the Hopf bundle, and it is called the Berry connection; its curvature is equal to the projection of the symplectic form in PH , modulo i . It may be written in a coordinate independent way as $A = i\langle\psi|d|\psi\rangle$.

We assume an arbitrary unitary time evolution $U(s)$ on the Hilbert space H , and we take an initial vector $|\psi_0\rangle$ at time $t = 0$. The curve

$$U(s)|\psi_0\rangle := |\psi(s)\rangle \quad (3)$$

projects to a curve $[\psi(s)]$ on the projective Hilbert space PH .

If we further assume that $U(s)$ is such that at time t , $[\psi(t)] = [\psi(0)]$, i.e. we have a loop γ on the projective space, then the phase that is transversed on the $U(1)$ fiber is equal to

$$e^{\int_0^t ds \langle\psi(s)| - \frac{d}{ds} - iH(s) |\psi(s)\rangle} := e^{iS[\psi(\cdot)]}, \quad (4)$$

where we wrote $H(s) = U^{-1}(s)\dot{U}(s)$ and S is the action out of which the Schrödinger equation is derived. The second term is a time dependent angle due to time evolution.

However, the first term is purely geometrical; it depends only on the transversed loop, and is equal to the holonomy of the Berry connection

$$e^{i\int_\gamma A} = \exp\left(-\int_\gamma \langle\psi|d\psi\rangle\right). \quad (5)$$

Note that the Berry phase does not change if we take different representatives $|\psi\rangle$ for the equivalence class $[\psi]$.

The geometric phase may also be defined for open paths by exploiting the metric structure on PH [16]. It allows us to form a loop from any path on the projective Hilbert space, by joining its endpoints with a geodesic. The geometric phase of the loop thus constructed is defined to be equal to the geometric phase associated to the open path. Hence if $\gamma = [\psi(\cdot)]$ is a path on PH , its associated geometric phase is proved to equal

$$e^{i\theta_g[\gamma]} = \exp\left(-\int_{t_i}^{t_f} dt \langle\psi(t)|\dot{\psi}(t)\rangle\right) \langle\psi_i|\psi_f\rangle. \quad (6)$$

This expression is meaningful only if the endpoints are not orthogonal.

Hence, the Berry phase is strongly related with geometric and topological structures of the Hilbert space of quantum theory. These geometric structures are physically relevant because of Born's probability interpretation: the single-time expectation values for observables do not change with phase transformations of the Hilbert space vector $|z\rangle \rightarrow e^{i\phi}|z\rangle$.

3 Histories

A history is defined as a sequence of projection operators $\alpha_{t_1}, \dots, \alpha_{t_n}$, and it corresponds to a time-ordered sequence of propositions about the physical system. The indices t_1, \dots, t_n *refer to the time a proposition is asserted and have no dynamical meaning*. Dynamics are related to the Hamiltonian H , which defines the one-parameter group of unitary operators $U(s) = e^{-iHs}$.

A natural way to represent the space of all histories is by defining a history Hilbert space $\mathcal{V} := \otimes_{t_i} \mathcal{H}_{t_i}$, where \mathcal{H}_{t_i} is a copy of the standard Hilbert space, indexed by the moment of time to which it corresponds. A history is then represented by a projection operator on \mathcal{V} . This construction has the merit of preserving the quantum logic structure [5] and highlighting the non-trivial temporal structure of histories [8, 9]. Furthermore, one can also construct a Hilbert space \mathcal{V} for continuous-time histories [6, 7, 17] by a suitable definition of the notion of the tensor product.

Furthermore, to each history α we may associate the class operator C_α defined by

$$C_\alpha = U^\dagger(t_n)\alpha_{t_n}U(t_n)\dots U^\dagger(t_1)\alpha_{t_1}U(t_1). \quad (7)$$

It is important to note that time appears in *two distinct places* in the definition of the class operator C_α : as the argument of the Heisenberg time evolution and as the parameter identifying the time at which a proposition is asserted. In what follows, we will show that this distinction is strongly related to the distinction between geometric and dynamical phase.

The decoherence functional is defined as a complex-valued function of pairs of histories: i.e. a map $d : \mathcal{V} \times \mathcal{V} \rightarrow \mathbf{C}$. For two histories α and α' it is given by

$$d(\alpha, \alpha') = \text{Tr} \left(C_\alpha \rho_0 C_{\alpha'}^\dagger \right) \quad (8)$$

The standard interpretation of this object is that when $d(\alpha, \alpha') = 0$ for $\alpha \neq \alpha'$ in an exhaustive and exclusive set of histories², then one may assign a probability distribution to this set as $p(\alpha) = d(\alpha, \alpha)$. The value of $d(\alpha, \beta)$ is, therefore, a measure of the degree of interference between the histories α and β .

4 The geometric phase for histories

We now consider a time interval $[t_0, t_f]$ and a history with $n + 1$ time steps $\alpha_{t_0}, \alpha_{t_1}, \dots, \alpha_{t_f}$. We assume that the projectors are fine-grained, which means that they correspond to elements of the projective Hilbert space

$$\alpha_{t_i} = |\psi_{t_i}\rangle\langle\psi_{t_i}|. \quad (9)$$

² By exhaustive we mean that at each moment of time t_i $\sum_{\alpha_{t_i}} \alpha_{t_i} = 1$ and by exclusive that $\alpha_{t_i}\beta_{t_i} = \delta_{\alpha\beta}$. Note that by α we denote both the proposition and the corresponding projector.

We first set the Hamiltonian equal to zero. The trace of the class operator C_α equals

$$TrC_\alpha = \langle \psi_{t_0} | \psi_{t_n} \rangle \langle \psi_{t_1} | \psi_{t_0} \rangle \langle \psi_{t_2} | \psi_{t_1} \rangle \dots \langle \psi_{t_n} | \psi_{t_{n-1}} \rangle \quad (10)$$

and it is non-zero provided there is no value of i , for which the vector $|\psi_{t_i}\rangle$ is orthogonal to $|\psi_{t_{i-1}}\rangle$.

Next, we assume that $\max |t_j - t_{j-1}| = \delta t$, and we choose the number of time steps n very large, so that $\delta t \sim O(n^{-1})$. Then $|\phi_{t_j}\rangle$ approximates a path $[\phi(t)]$ on PH . Hence,

$$\begin{aligned} \log TrC_\alpha &= \log \langle \psi_{t_0} | \psi_{t_n} \rangle + \sum_{i=1}^n \log \langle \psi_{t_i} | \psi_{t_{i-1}} \rangle \\ &= \log \langle \psi_{t_0} | \psi_{t_n} \rangle + \sum_{i=1}^n \log (1 - \langle \psi_{t_i} | \psi_{t_i} - \psi_{t_{i-1}} \rangle) \end{aligned} \quad (11)$$

and the limit of large n yields

$$\log TrC_\alpha = \log \langle \psi_{t_0} | \psi_{t_n} \rangle - \langle \psi_{t_i} | \psi_{t_i} - \psi_{t_{i-1}} \rangle + O((\delta t)^2) \quad (12)$$

As $\delta t \rightarrow 0$ the sum in the right-hand side converges to a Stieljes integral $-\int_{t_i}^{t_f} dt \langle \psi(t) | \dot{\psi}(t) \rangle$. Hence for a continuous path we take

$$TrC_\alpha = e^{i\theta_g[\psi(\cdot)]} \quad (13)$$

Therefore, the map $\alpha \rightarrow TrC_\alpha$ assigns to each fine-grained “continuous-time” history α its corresponding Berry phase. In fact, the paths $\psi(\cdot)$ need not be continuous; it suffices that the Stieljes integral is defined.

Furthermore, one may use the above result to define the Berry phase, associated to a general coarse-grained history. Hence, if $\alpha = (\hat{\alpha}_{t_1}, \dots, \hat{\alpha}_{t_n})$ is a history, then we may write

$$\alpha \rightarrow Tr(\hat{\alpha}_{t_n} \dots \hat{\alpha}_{t_2} \hat{\alpha}_{t_1}). \quad (14)$$

This defines a map from \mathcal{V} to the complex numbers, that *assigns to each history its corresponding geometric phase*. In particular, if we decompose the projector $\hat{\alpha}_{t_i}$ with respect to an orthonormal basis in the subspace, in which it projects

$$\hat{\alpha}_{t_i} = \sum_r |\psi_{t_i}^r\rangle \langle \psi_{t_i}^r|, \quad (15)$$

we may then write the geometric phase for the coarse-grained histories as

$$\sum_{r_1, \dots, r_n} e^{i\theta_g[\psi_{r_1 \dots r_n}(\cdot)]} \quad (16)$$

In the continuum limit this can be written, suggestively, as a sum over all fine-grained paths $\psi(\cdot)$ compatible with the coarse-grained history α

$$\sum_{\psi(\cdot) \in \alpha} e^{i\theta_g[\psi(\cdot)]} \quad (17)$$

In view of this linearity, the map that assigns to each history the corresponding Berry phase can be described by a functional on \mathcal{V} . When \mathcal{V} with a tensor product of single-time Hilbert spaces, this linear functional is naturally induced by the tensor product construction.

We must note here that, our definition of the geometric phase is structurally distinct from the standard one. The latter refers to the evolution of a *state* under a dynamical law. In histories formalism, the geometric phase is defined on *observables*, or, more precisely, on *possible* scenaria for the physical system. There is, therefore, no need to make any assumption about the dynamics: this definition of geometric phase makes sense even if the dynamics is non-unitary.

5 The structure of the decoherence functional

The standard form of the decoherence functional incorporates the histories α by means of the operator \hat{C}_α . This suggests an expression for the decoherence functional that can be written in terms of the geometric phase.

To this end, let us assume two “continuous-time” histories, which we shall denote as $\alpha_{\phi(\cdot)}$ and $\alpha_{\psi(\cdot)}$. From Eq. (13), and for the decoherence functional written for vanishing Hamiltonian, we take

$$\begin{aligned} d(\alpha_{\psi(\cdot)}, \alpha_{\phi(\cdot)}) &= \\ &= \langle \phi(t_i) | \rho_0 | \psi(t_i) \rangle \langle \psi(t_f) | \phi(t_f) \rangle \times \exp \left(- \int_{t_i}^{t_f} dt \langle \psi(t) | \dot{\psi}(t) \rangle + \int_{t_i}^{t_f} dt \langle \dot{\phi}(t) | \phi(t) \rangle \right) \end{aligned} \quad (18)$$

The two histories form a loop on PH , provided that their endpoints coincide. For example, this is the case where the density matrix ρ_0 is pure, and hence equal to an one-dimensional projector that could be considered as part of the history. From Eq. (14) we conclude that, the value of the decoherence functional is the Berry phase, associated to this loop.

When the Hamiltonian is included, we find

$$d(\alpha_{\psi(\cdot)}, \alpha_{\phi(\cdot)}) = \langle \phi(t_i) | \rho_0 | \psi(t_i) \rangle \langle \psi(t_f) | \rho_f | \phi(t_f) \rangle e^{iS[\psi(\cdot)] - iS^*[\phi(\cdot)]}, \quad (19)$$

where the action operator S [8] is given by the expression

$$S[\phi(\cdot)] = \int_{t_i}^{t_f} dt \langle \phi(t) | i \frac{d}{dt} - H | \phi(t) \rangle \quad (20)$$

Hence the phase change on the Hopf bundle enters the decoherence functional at the level of the most general fine-grained histories.

Let us now note the following:

First, the appearance of the action is contingent upon the dynamics given by a Hamiltonian. One may consider more general dynamics: they are incorporated in the decoherence functional through the map $\hat{\alpha}_t \rightarrow \hat{\alpha}_t(t)$ that assigns to each Schrödinger-picture projector $\hat{\alpha}_t$, a corresponding Heisenberg-picture one $\hat{\alpha}_t(t)$, at time t . In full generality, it suffices that the dynamics is generated by an one-parameter family of automorphisms of the algebra of operators on the Hilbert space \mathcal{H} (it does not even need to be an one-parameter group). Hence, even though the expression involving the action is suggestive and simple, it is not as fundamental and general as Eq. (18), which expresses the decoherence functional in terms of the Berry phase, prior to the introduction of the dynamics. One should keep in mind that one aim of the histories programme is to describe physical systems that have non-trivial temporal structure—as arising, for instance, in quantum gravity—and are, perhaps, not amenable to a Hamiltonian description. The equation (18) for the decoherence functional is of sufficient generality to persist even in such contexts.

Second, following our earlier reasoning, it is easy to show that the fine-grained expressions for the decoherence functional can be used to determine its values for general coarse-grained histories. In analogy to (17) they read

$$d(\alpha, \beta) = \sum_{\psi(\cdot) \in \alpha} \sum_{\phi(\cdot) \in \beta} \langle \phi(t_i) | \rho_0 | \psi(t_i) \rangle \langle \psi(t_f) | \rho_f | \phi(t_f) \rangle e^{iS[\psi(\cdot)] - iS^*[\phi(\cdot)]} \quad (21)$$

Finally, the knowledge of the geometric phase—for a set of histories and of the automorphism that implements the dynamics—is sufficient to fully reconstruct the decoherence functional – and hence all the probabilistic content of a theory. The contribution of the initial state can be obtained by convex combinations of a pure state at the initial moment of time. What is interesting, is that at this level *there is no need for our system to be described by a Hilbert space*. All that is needed is a space of paths—on any manifold, the $U(1)$ connection from which the functional giving the Berry phase will be constructed and the dynamical law in the form of an automorphism of the space of observables. This can be an important starting point for developing geometric procedures for *quantisation* of quantum mechanical histories.

6 Conclusions

From Eq. (13) we notice that the Berry phase arises solely from the *ordering in time* of the projection operators, as they appear in the decoherence functional. It eventually corresponds to the kinematical part of the action Eq. (20). The Hamiltonian part appears due to Heisenberg-type time evolution of the projectors. This distinction is a fundamental and impressive feature of history theories

that was identified in [8, 9]. There exist two distinct ways, in which time appears in physical theories: as a distinction between past and future (partial ordering property of time) and as the parameter underlying the evolution laws (time as parameter of change).

One of us (N.S.) has shown that these notions of time are associated to the kinematical and dynamical part of the action functional respectively, and there exist distinct operators that generate time translation with respect to these two parameters. They are an irreducible part of any theory that is based on temporally extended objects, whether classical or quantum. This distinction is manifested in the two different ways the time parameter appears in the definition of the class operator C_α . From Eqs. (13–20) we see that this is *identical* to the distinction between the geometric and the dynamical phases of standard quantum theory. In a sense, this is the only non-trivial remnant in canonical quantum theory of the temporal structure of history theories. The reader is referred to [9] for a fuller treatment of this issue and to [18] for the merits of the quantisation scheme motivated by it.

The fact that the off-diagonal elements of the decoherence functional correspond to the difference in Berry phase between its histories, suggests that the current interpretation of probabilities in the consistent histories scheme is at least incomplete. The relative geometric phase between two histories is a measurable quantity, while the present interpretation gives physical meaning to the values of only the *diagonal elements* of the decoherence functional.

Of course, one might argue that the geometric phase is measured only by comparing statistical measurements in two different *ensembles* of systems. As such, it may be described as any other measurement in the scheme. However, the point we make is that, the off-diagonal elements of the decoherence functional have a clear geometric and operational meaning. Therefore, an interpretational scheme that ignores them might face a truncation of the physics it addresses. In addition, the Berry phase would constitute a quantity that cannot be explained in terms of the properties of an individual quantum system, even though it is *measured* in ensembles. This is extremely problematic for the aims of a realist interpretation of quantum theory.

Our results highlight the presence of the complex phases in time evolution *at the purely kinematical level*, as the main contributors in the non-additivity of the probability measure for histories.

This strongly suggests that the presence of complex numbers in quantum theory is intrinsically linked to its *distinct* “probabilistic” structure. To see this consider the following.

First, both classical and quantum probability theory at a single moment of time are described by an additive measure over a lattice of propositions. But when time-evolution takes place in quantum theory, there appear complex phases that render the probability measure non-additive (this is the essence of the interference of histories).

Second, the pure time evolution in standard quantum theory is of a Hamilto-

nian type on PH ; the dynamical phases that are generated by the Hamiltonian, are *structurally not different* from any angle variables of classical mechanics. There is *nothing inherently complex in them*, as the Schrödinger equation can be written without any reference to an i . On the other hand the geometric phase appears due to the bundle structure of the quantum mechanical space of rays. The bundle structure arises in the first place, because single-time probabilities do not depend on phase. Hence, even in standard quantum theory there is an indirect relation between the Berry phase and the probability assignment. This is brought fully into focus in the histories formalism.

We explained in the introduction, that when we are restricted in a single moment of time, the structures of quantum theory are in one-to-one analogy with the ones of classical probability theory. From an operational perspective, quantum mechanics at a single moment of time may be formulated without making any reference to complex numbers; it can be stated solely in terms of real-valued observables, expectation values and probabilities. It is only, when we study physical systems in a temporal sense that complex numbers appear. However, their appearance cannot be attributed to the law of time evolution.

Dynamics, in general, appears as an automorphism of the space of observables: if the observables are defined as real-valued, they will remain real-valued when dynamically transformed. For example, Schrödinger's equation does not need introduce the complex unit; it can be equally well written in a real Hilbert space [19].

Alternatively, one may substitute Schrödinger's equation with a real, partial differential equation on phase space—using either the Wigner or the coherent state transforms. Hence, while complex numbers in quantum theory are unavoidable when we study properties of the system at different moment of time, *they are not introduced by the dynamical law*. Furthermore, it is the temporal ordering that introduces phases, in an irreducible way, into the decoherence functional, that it is encoded in the definition of the class operator C_α .

In other words, the geometric phase is a genuinely complex-valued object; and it is only the fact that we *measure* such an object, that *forces* us to accept complex numbers as an irreducible part of quantum theory. Complex number are not a necessary consequence of *any dynamical law*.

Hence, we conclude that, *the complex structure of quantum theory is intrinsically linked to both its probability structure and the way the notion of succession is encoded*. After all quantum theory is a theory of amplitudes, and the results from the above analysis imply that *all physically relevant amplitudes*—contained in the decoherence functional—are constructed from the geometric phase. As such they are *geometrical in origin*.

This is an intriguing result. It is a structural characteristic of quantum probability that should persist in frameworks that attempt to generalise quantum theory in a way that the Hilbert space is not a necessary ingredient.

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